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POTENTIAL FOR CURVE CRACK IN ELASTICITY

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Abstract
In this paper we are discussed four types potential in antiplane elasticity these are point dislocation, concentrated force, dislocation doublet and concentrated force doublet. It is show that if the axis of the concentrated force doublet is orthogonal to the direction of the dislocation doublet, the relevant potentials are equal. Using the obtained potentials, an integral equation for the curve crack problem is introduced. Some particular features of the obtained singular integral equation are discussed and numerical solutions and example are given.
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Keywords: Potentials in antiplane elasticity; Curve crack problem; Numerical solution of integral equation.

1. Introduction:
After distributing density of the appropriate potential along the curve, one can model the curve crack problem in antiplane elasticity, which results in a hyper singular integral equation [6]. According to [2, 3], placing a particular force doublet along a line, the crack problem in plane and antiplane elasticity can be solved. However, less attention was paid to the curve crack problem in [2, 3]. In this paper potentials and singular integral equation suitable for solving curve crack problem are developed by using the Cauchy type integral and Sokhotskii-Plemelj formula [4]. A numerical technique is suggested to solve the obtained singular integral equations. Four types of the potential in antiplane elasticity are defined in the whole plane with the singular point $z = a$. We are derived from point dislocation at point $z = a$, concentrated force at point $z = a$, dislocation doublet at point $z = a$ with intensity H and the direction α and concentrated force doublet at point $z = a$ with intensity B and direction β.
It is prove that if the direction of the concentrated force doublet is normally interacted to the direction of the dislocation doublet, the relevant potentials are equivalent. Using the obtained result, the distributed dislocation doublet can model the crack problem in antiplane elasticity more directly.

2. Fundamental potentials in elasticity:
Potential in antiplane elasticity can be expressed

$$\phi(z) = G\omega(x, y) \quad (1)$$

$$\phi'(z) = \sigma_{xz} - \sigma_{yz} = G\left(\frac{\partial \omega}{\partial x} - \frac{\partial \omega}{\partial y}\right) = \frac{\partial f}{\partial y} + \frac{\partial f}{\partial x} \quad (2)$$

Where G is the shear modulus of elasticity, ω the out of plane displacement, f the resultant force function, and σ_{xz}, σ_{yz} the stress components . Equilibrium equation takes the form

$$\frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial x} = 0 \tag{3}$$

and the resultant force

$$f(x,y) = \int_{z_0}^{z} \sigma_{xz} dy - \sigma_{yz} dx \tag{4}$$

In (4), the domain of integration is a path connecting the fixed point $z_0 = a_0$ and the generic point $z = a$. On the basis of the equilibrium Eq. (3), the integral (4) is path independent.

In the following analysis, four potentials are introduced: point dislocation, concentrated force, dislocation doublet and concentrated force doublet (Fig. 1). All of them take the point at $z = a$.

(a) The potential for the point dislocation at $z = a$ is (Fig. 1(a))

$$\phi(z) = E \log(z - a) \tag{5}$$

E is the intensity of the point dislocation. Let $\{g\}_a$ denote the contour increment of a function $g(x,y)$ for the closed path around the point $z = a$.in Fig. 1(a), from (1) and (5), we have

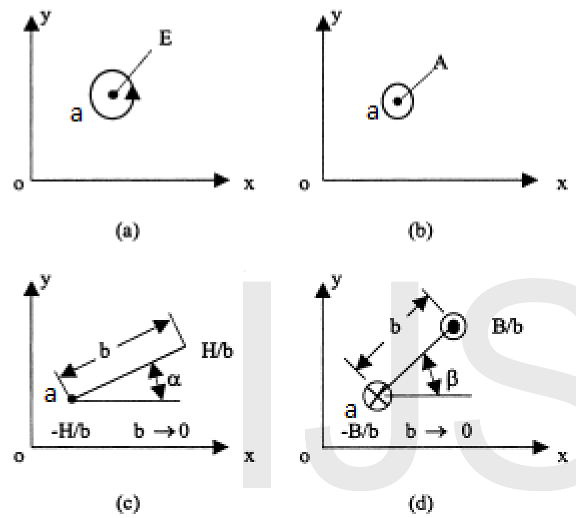

Fig. 1. (a) A point dislocation at the point $z = a$. (b) A concentrated force at the point $z = a$. (c) A dislocation doublet at the point $z = a$.with the intensity H and direction α and (d) A concentrated force doublet at the point $z = a$.with the intensity B and direction β.

$$\{\phi\}_a = \{G\omega\}_a = -2\pi E, \{f\}_a = 0 \tag{6}$$

(b) The potential for the concentrated force at $z = a$ is (Fig. 1(b))

$$\phi(z) = A \log(z - a)$$

From (1) and (7), we have (7)

$$\{\phi\}_a = \{f\}_a = 2\pi A,\ \{G\omega\}_a = 0 \tag{8}$$

(c) The potential for the dislocation doublet can be obtained by superimposing point dislocations with intensity $-\frac{H}{b}$ and $\frac{H}{b}$ at the point $z = a$ and $z = a + be^{\alpha}$. Respectively (Fig. 1(c)), and letting $b \to 0$

$$\phi(z) = \lim_{b \to 0} \frac{H}{b}\left[\log\left(z - (a + be^{\alpha})\right) - \log(z - a)\right] = \frac{H}{a - z} e^{\alpha} \tag{9}$$

This potential depends on two parameters, the intensity H and the angular value α indicating the dislocation doublet axis (Fig. 1(c)).

(d) The potential for the concentrated force doublet can be obtained in a similar manner. A couple of forces with intensity $-\frac{B}{b}$ and $\frac{B}{b}$ are placed at the $z = a$ and $z = a + be^{\beta}$ respectively (Fig. 1(d)), and the limit for $b \to 0$ is calculated

$$\phi(z) = \lim_{b\to 0} \frac{B}{b}\left[\log\left(z - \left(a + be^{\beta}\right)\right) - \log(z - a)\right] = \frac{B}{a-z} e^{\beta} \tag{10}$$

This potential depends on two parameters, the intensity B, and the angular value β indicating the dislocation doublet axis (Fig. 1(d)).

Let us consider Eqs. (9) and (10), by assuming $B = H$ and $\beta = \alpha + \frac{\pi}{2}$, a direct substitution demonstrates the equivalence of the potentials (c) and (d). This indicates that a dislocation doublet and a force doublet with the same intensity and mutually orthogonal directions produce the same stress field.

3. Solution for curve crack problem:

By placing a continuous distribution of dislocation doublet along the curve L (Fig. 2), from (9) we can obtain the potential as follows:

$$\phi(z) = \frac{1}{2\pi}\int_L \frac{h(t)dt}{a-z} \tag{11}$$

Physically, the potential represents the continuous distribution of dislocation doublet along the curve L, where the axis of doublet coincides with tangent of the curve (Fig. 2). In fact, since $dt = ds\, e^{\alpha}$ as shown in Fig. 2, the integrand in (11) can be rewritten as

$$\frac{1}{2\pi}\frac{h(t)dt}{a-z} = \frac{h(t)ds}{2\pi}\frac{1}{a-z}e^{\alpha} \tag{12}$$

Which is the form of the potential shown in (9).

In order to study the behavior of the relevant functions, we make the following definition:

$$[g(a)] = g^{+}(a) - g^{-}(a)\,,\ (a \in L) \tag{13}$$

Where the superscript $+(-)$ means the upper (lower) limit of the function g approaching the curve L (Fig. 2). By the use of the Sokhotskii-Plemelj formula [4], from (11) we arrive at

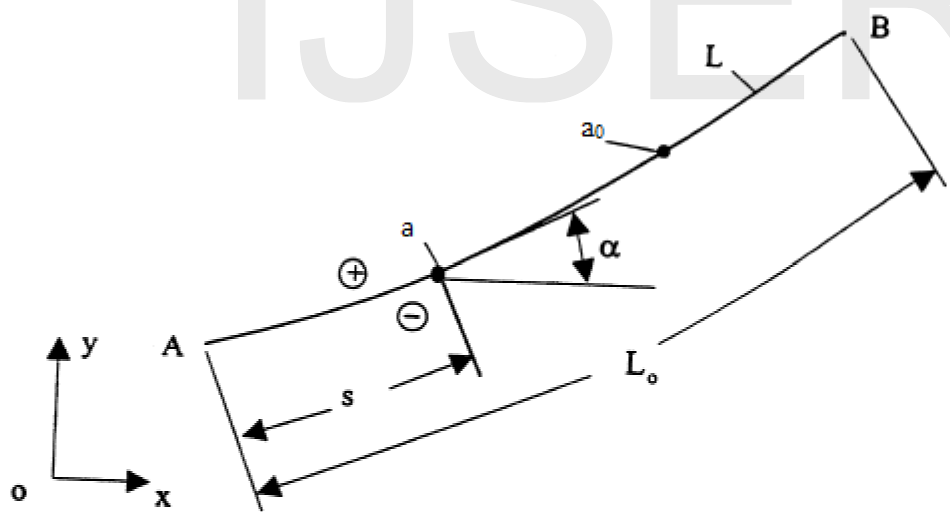

Fig. 2. Distribution of the dislocation doublet along a curve L (a value representation of a point on the Curve $L, s -$ the relevant arc length coordinate).

$$[\phi(a)] = [G(\omega(a))] = L(a),\ [f(a)] = 0\ ,(a \in L) \tag{14}$$

This equation reveals that, when a moving point is normally passing through the curve L, the displacement component ω is discontinuous, and the resultant force function is continuous, thus suggesting that the potential (11) can model the curve crack problem in antiplane elasticity.

In the crack problem, we assume the stresses vanish and some traction is applied along the crack face. The resultant force function defined along the curve is denoted by $f(a)\ (a \in L)$. which can be evaluated from the traction applied to the crack. Letting $z \to a_0^{+}$ or $z \to a_0^{-}$ (Fig. 2), in both case we obtain the following singular integral equation:

$$\int_L \frac{h(t)dt}{a-a_0} = -2\pi f(a_0) + e_1 \quad [a_0 \in L] \tag{15}$$

Where e_1 is a constant.

In fact, in this case the original field can be decomposed into a uniform field defined by an infinite body without crack and a local field. By using (2), the resultant force function for the uniform field can be written as

$$f_u = -\sigma_{yz}^{\infty} x + \sigma_{xz}^{\infty} y - e_1 \tag{16}$$

Where the subscript u denotes the uniform field solution and e_1 is a constant.

For the local field, the remote stresses vanish and the tractions applied on the crack face are opposite to those acting in the same position in the uniform field. Therefore

$$f(a_0) + e_1 = -f_u(a_0) = \sigma_{yz}^{\infty} x_0 - \sigma_{xz}^{\infty} y_0 + e_1 \quad [a_0 \in L] \tag{17}$$

For convenience, we write the function $h(t)|_{t=t(s)}$ simply as $h(s)$ which represents the crack opening displacement (COD). In antiplane elastic crack problem, at the vicinity of the left crack tip $h(s) = O\left(s^{\frac{1}{2}}\right)$ (Fig. 2), therefore, the mode III stress intensity factors at the crack tips A and B can be obtained respectively by

$$K_{3A} = \frac{1}{2}\sqrt{\frac{\pi}{2}} \lim_{s\to 0} h(s) s^{-\frac{1}{2}} \tag{18}$$

$$K_{3B} = \frac{1}{2}\sqrt{\frac{\pi}{2}} \lim_{s\to 0} h(s)(L_0 - s)^{-\frac{1}{2}} \tag{19}$$

L_0 being the length of the crack (Fig.2).

4. Numerical solution and examples:

In order to get a numerical solution, the curve crack can be approximated by a suitable sequence of N segments connecting the points $P_0, P_1, \dots, P_{j-1}, P_j, \dots, P_{N-1}, P_N$ (Fig. 3). In the local coordinates having origins in the segment midpoints the $h(s)$ function can be approximated in the form [1] (Fig. 3).

$$h(s_1) = c_1\sqrt{\frac{1}{2}\left(1 + \frac{s_1}{d_1}\right)} \quad , |s_1| \le d_1 \tag{20}$$

$$h(s_j) = \frac{c_{j-1}}{2}\left(1 - \frac{s_j}{d_j}\right) + \frac{c_j}{2}\left(1 + \frac{s_j}{d_j}\right) \; , \; |s_j| \le d_j \; , \; j = 2,3,\dots,N-1 \tag{21}$$

$$h(s_N) = c_{N-1}\sqrt{\frac{1}{2}\left(1 + \frac{s_N}{d_N}\right)} \quad , |s_N| \le d_N \tag{22}$$

Where $c_1, c_2, \dots, c_{N-2}, c_{N-1}$ represent the COD values at the nodes $P_1, P_2, \dots, P_{N-2}, P_{N-1}$ respectively.

In the numerical solution, the integral condition (15) is imposed at the midpoint of any interval, $\phi_1, \phi_2, \dots, \phi_{N-1}, \phi_N$ (Fig. 3). This leads to the following integrals:

$$I_1 = \int_0^1 \frac{\sqrt{1+x}}{x}\, dx = \sqrt{2} - \log\left(\sqrt{2} + 1\right) \quad , \quad I_2 = \int_0^1 \frac{dx}{x} = 0 \tag{23}$$

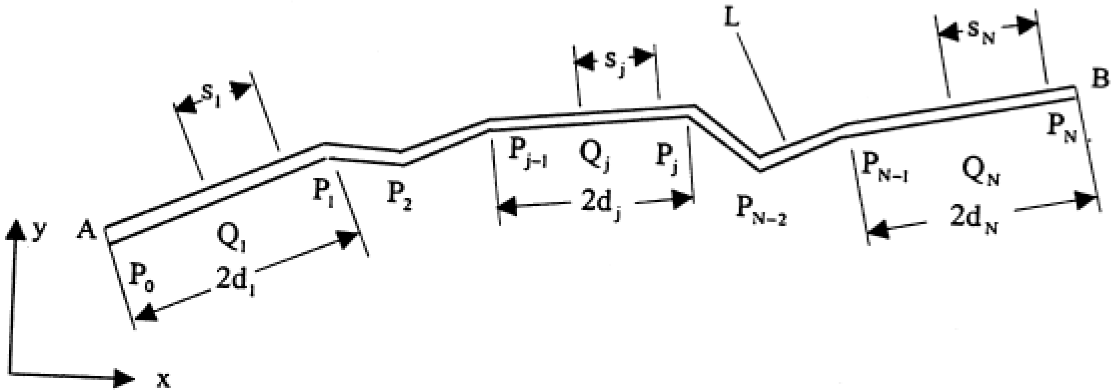

Fig. 3. A winding crack consisting of N segments.

$I_3 = \int_0^1 \frac{g(x)dx}{x-p}$, $P \notin [0,1]$ (24)

The integral (25) being$g(x)$ a regular function can be obtained by a numerical quadrature rule. The integral equation (15) is then reduced into the following system of algebraic equations:

$\sum_{k=1}^{N-1} D_{jk}\, C_k = F_j + e_1$ $(j = 1,2, \dots, N)$ (25)

Where the coefficients D_{jk} $(j = 1,2, \dots, N\,, k = 1,2, \dots, N-1)$ can be obtained from the assumed discretization shown by (20)-(22), andF_j $(j = 1,2, \dots, N)$ can be obtained from the given boundary value at the point Q_j(Fig. 3). In the present case, σ_{xz}^{∞} vanishes, and from (17)

$f(a_0) = \sigma_{yz}^{\infty} x_0$, thus, substituting the coordinate x_0 for the point Q_j , the F_j value is obtainable. By subtracting the Nth equation from the jth equation $(j = 1,2, \dots, N-1)$.

$\sum_{k=1}^{N-1} (D_{jk} - D_{Nk}) C_k = F_j - F_N$ $(j = 1,2, \dots, N-1)$ (26)

the constant e_1 disappears, and $N-1$ equations for determining the $N-1$ unknowns $C_1, C_2, \dots, C_{N-2}, C_{N-1}$ are obtained.

On the basis of the obtained CODs: $C_1, C_2, \dots, C_{N-2}, C_{N-1}$ and Eqs. (18) and (19), the stress intensity factors at the tips A and B can be evaluated.

$K_{3A} = \frac{1}{4}\sqrt{\frac{\pi}{d_1}}\, C_1\, , K_{3B} = \frac{1}{4}\sqrt{\frac{\pi}{d_N}}\, C_{N-1}$ (27)

In the following numerical examples the cracks were divided into $N = 80$ equal segments. For an infinite body carrying a straight crack with length 2ι loaded by the remote stress σ_{yz}^{∞} , the result is $K_3 = 1.0002\sigma_{yz}^{\infty}\sqrt{\pi\iota}$, which is very close to the exact solution $K_3 = \sigma_{yz}^{\infty}\sqrt{\pi\iota}$ expressed is [10] .

For a bent crack under the remote stress $\sigma_{yz}^{\infty}$with an angle h . 45° (Fig. 4), the results, expressed as

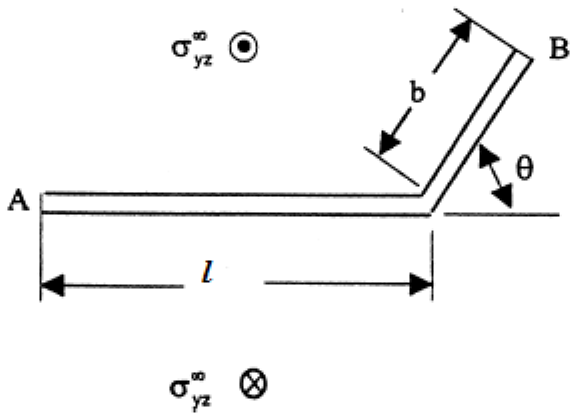

Fig. 4 A bent crack configuration.

$$K_{3A} = F_A\left(\frac{b}{l}\right)\sigma_{yz}^{\infty}\sqrt{\frac{\pi(l+b\cos\theta)}{2}}$$

$$K_{3B} = F_B\left(\frac{b}{l}\right)\sigma_{yz}^{\infty}\sqrt{\frac{\pi(l+b\cos\theta)}{2}} \qquad (28)$$

are listed in Table 1 and compared with [9].

The angle θ was then changed in the interval $[0° - 165°]$ for several ratios $\frac{b}{l+b}$ The calculated results are expressed as

$$K_{3A} = G_A\left(\frac{b}{l+b},\theta\right)\sigma_{yz}^{\infty}\sqrt{\frac{\pi(l+b)}{2}}$$

$$K_{3B} = G_B\left(\frac{b}{l+b},\theta\right)\sigma_{yz}^{\infty}\sqrt{\frac{\pi(l+b)}{2}} \qquad (29)$$

Comparison of normalized stress intensity factors $F_A\left(\frac{b}{l}\right)$, $F_B\left(\frac{b}{l}\right)$ for the bent crack (see Fig. 4 and Eq. (28))

$\frac{b}{l}$	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
F_A^l	1.015	1.013	1.119	1.025	1.129	1.032	1.035	1.137	1.039	1.140
F_A^b	1.008	1.014	1.020	1.025	1.028	1.031	1.034	1.036	1.037	1.038
F_B^l	0.915	0.925	0.869	0.875	0.868	0.860	0.853	0.839	0.845	0.842
F_B^b	0.916	0.898	0.884	0.873	0.864	0.856	0.850	0.845	0.841	0.837

l This paper.
b In Ref. [3].

REMARKS:

Four types of potential obtained by placing some singular source at the point $z = a$, are introduced solving for antiplane curve crack problem. The COD value derive directly from the solution of the singular equation.

REFERENCES:

[1] H. Nisitani, D.H. Chen, Body Force Method, Zhejiang University Press, Hangzhou, 1990 (Chinese translation from Japanese).
[2] H. Nisitani, Solution of notch problems by body force method, in: G.C. Sih (Ed.), Mechanics of Fracture, vol. 5, Noordhoff, Leyden, 1978, pp. 1-68.
[3] G.C. Sih, ASME Journal of Applied Mechanics 32 (1965) 51.
[4] N.I. Muskhelishvili, Some Basic Problems of Mathematical Theory of Elasticity, Noordhoff, Groningen, 1953.
[5] N.I. Muskhelishvili, Singular Integral Equation, Noordhoff, Groningen, 1953.
[6] Y.Z. Chen, N. Hasebe, ASME Journal of Applied Mechanics 63 (1996) 845.
[7] Y.Z. Chen, Z.X. Wang, International Journal of Fracture 30 (1986) 287.
[8] Y.Z. Chen, International Journal of Fracture 64 (1993) 63.
[9] Y.K. Cheung, Y.Z. Chen, International Journal of Fracture 65 (1994) 359.
[10] Y.Z. Chen, International Journal of Engineering Science 38(200) 565-574.